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Ballistic Entry Motion, Including Gravity: Constant Drag Coefficient Case

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Existing analytic theories for the motion of a ballistic nonlifting entry vehicle cannot accurately predict large reductions in vehicle flight-path angle that can occur for large drag coefficients. A new semianalytic solution, accurate for many entry conditions, is derived using asymptotic matching concepts. Solutions for the velocity components as functions of altitude are reduced to quadratures involving the atmospheric density profile and a *constant* drag coefficient. Numerical results are computed using a recent air density model.

Background

An exact analytic solution for the trajectory of a lifting entry vehicle is not known to exist. Approximate analytic theories are numerous.[1-7] For example, the flight-path angle is constant if gravity and aerodynamic lift are neglected. It has been shown that this simplification permits a closed-form solution for velocity as a function of altitude in the cases of constant drag coefficient[1] and a certain integrable velocity-dependent drag coefficient.[2]

The method of matched asymptotic expansions[8] has been used extensively in entry flight dynamics problems.[9-15] Approximate trajectories have been derived by matching an exoatmospheric Keplerian solution with a rectilinear endoatmospheric solution. Asymptotic matching techniques have also been applied to problems involving decay of a circular orbit[12,13] and optimal control of lifting vehicles.[14,15]

Approach

Forces affecting the endoatmospheric trajectory of a nonlifting vehicle arise from gravity and aerodynamic drag. The drag vector is opposite to the vehicle velocity relative to the atmosphere, which rotates with the Earth. Close to the Earth, a flat Earth simplification is accurate. The gravity field is approximately uniform with constant acceleration g. Although a flat Earth is assumed, atmospheric motion is modeled as a constant wind vector (speed 465 m/s) in the horizontal plane.

In the absence of lateral forces produced by aerodynamic lift and local wind perturbations, motion occurs in the spatially invariant (vertical) plane containing the gravity and drag vectors. The flat Earth equations of motion are

$$\frac{\mathrm{d}v}{\mathrm{d}t} = w\frac{\mathrm{d}v}{\mathrm{d}h} = -\tfrac{1}{2}k_\infty \rho(h)\sqrt{v^2+w^2}\,v \tag{1a}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w\frac{\mathrm{d}w}{\mathrm{d}h} = -\tfrac{1}{2}k_\infty \rho(h)\sqrt{v^2+w^2}\,w - g \tag{1b}$$

Altitude h is chosen as the independent variable since air density ρ depends on h. The vehicle velocity relative to a coordinate system translating (at constant speed) with the atmosphere is resolved into horizontal v and vertical w components in the invariant trajectory plane. The constant parameter k_∞ is defined as the product of the hypersonic inviscid drag coefficient at zero angle of attack, and the reference area-to-mass ratio.

The equations of motion (1) will be solved for v and w as functions of h. Time t and downrange position x in the invariant trajectory plane may be obtained from the following quadratures:

$$t = \int_H^h \frac{\mathrm{d}h}{w} \tag{2a}$$

$$x = \int_H^h (v/w)\,\mathrm{d}h \tag{2b}$$

where H is the entry point altitude.

Qualitatively, solutions plotted in (w,v) velocity space (Fig. 1) asymptotically approach the w axis. The slope $\mathrm{d}w/\mathrm{d}v$ of a velocity space trajectory is

$$\frac{\mathrm{d}w}{\mathrm{d}v} = \frac{\dot{w}}{\dot{v}} = \frac{w}{v} + f(v,w,h)\frac{\sqrt{v^2+w^2}}{v} \tag{3a}$$

$$f(v,w,h) = \frac{g}{\tfrac{1}{2}k_\infty \rho(h)\,(v^2+w^2)} \tag{3b}$$

where f is the ratio of gravitational acceleration to drag deceleration. Endoatmospheric trajectories consist of three approximately linear segments, corresponding to case 1: high altitude entry ($\rho \simeq 0$); case 2: $f \ll 1$; and case 3: vertical descent ($v=0$). Case 3 occurs if, for example, k_∞ is sufficiently large, or if $v=0$ at the entry point. Nonlinear transitions between linear segments occur when $f \cong 1$.

An exact analytic solution, incorporating all phases of entry motion (Fig. 1), is not known to exist. Three well-known approximate solutions exist. In configuration space with coordinates (x,h), cases 1 and 2 consist respectively of a parabolic trajectory perturbed by drag and a rectilinear trajectory perturbed by gravity. It has been demonstrated that asymptotic matching techniques can be used to combine these approximate solutions in a single, uniformly valid solution.[9-11] Case 3 has a well-known closed-form solution consisting of a vertical rectilinear trajectory.[1]

A new semianalytic solution will combine cases 1-3. An outer solution incorporates cases 1 and 2, and is accurate during the initial entry phase, typically until peak deceleration occurs. In the limit of zero drag coefficient, the outer solution reduces to the exact exoatmospheric solution (a parabola). The inner solution is needed during the terminal entry phase, from peak deceleration to impact, in order to predict the transition to case 3. The inner and outer solutions are combined using asymptotic matching techniques.

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The Outer Solution

The outer solution will be derived using nondimensional variables ω, λ, and $\epsilon\zeta$, defined by

$$\omega=(v/V)^2 \qquad \lambda=(v/w)^2 \qquad \zeta=h/H \tag{4}$$

V is the horizontal component of relative velocity at the entry point altitude H. It is clear that $0\le\zeta\le 1$ and $0\le\omega\le 1$ during entry.

Using the new variables, it can be shown that the differential equations (1) are equivalent to

$$\frac{d\omega}{d\sigma}+\sqrt{1+\lambda}\,\omega=0 \tag{5a}$$

$$\omega\frac{d}{d\zeta}\left(\frac{1}{\lambda}\right)+2\epsilon=0 \tag{5b}$$

$$\epsilon=gH/V^2 \tag{5c}$$

The transformation equation (4) introduces a small parameter ϵ, which is the ratio of (flat Earth) gravitational potential energy to twice the entry point horizontal kinetic energy. The constant k_∞ is included in the nondimensional air density quadrature σ, defined by

$$\sigma=k_\infty\int_h^H \rho\,dh \tag{6}$$

which is a closed-form function of altitude for integrable density profiles. For example, σ increases exponentially for exponential density profiles. Consequently, if $\omega\sim 0(1)$, ω will decrease at a faster rate compared to λ as altitude decreases.

The solutions of the differential equations (5) may be written in the form

$$\omega=e^{-\phi} \tag{7a}$$

$$\lambda=b(1+2\epsilon b\psi)^{-1} \tag{7b}$$

The quadratures ϕ and ψ satisfy a set of simultaneous integral equations:

$$\phi=\int_0^\sigma\sqrt{1+\lambda}\,d\sigma=\int_0^\sigma\sqrt{1+b(1+2\epsilon b\psi)^{-1}}\,d\sigma \tag{8}$$

$$\psi=\int_\zeta^1\frac{d\zeta}{\omega}=\int_\zeta^1 e^{\phi}d\zeta \tag{9}$$

The initial conditions at the entry point ($\zeta=1$) are $\omega=1$ and $\lambda=b$, where the constant b is the square of the cotangent of the initial flight-path angle. Although customary to expand ϕ and ψ in asymptotic series, a more accurate procedure will be used in the sequel.

The method of successive approximations will be used to solve for ϕ and ψ, which, as will subsequently be shown, may be expressed in terms of the constants b, k_∞, ϵ, and certain integrals involving the air density profile. If $\epsilon=0$, the integral equations uncouple, yielding zeroth-order (approximate) solutions ϕ_0 and ψ_0:

$$\phi_0=\int_0^\sigma\sqrt{1+b}\,d\sigma=\sqrt{1+b}\,\sigma \tag{10a}$$

$$\psi_0=\int_\zeta^1 e^{\phi_0}d\zeta=\int_\zeta^1 e^{\sqrt{1+b}\sigma}d\zeta \tag{10b}$$

The quadrature ϕ_0, since it is proportional to σ, is a closed-form function of altitude. Although its integrand may also be computed analytically, ψ_0 must be evaluated using a numerical quadrature algorithm. First-order changes in λ (since $\psi_0\ne$ const) are incorporated in ϕ_1:

$$\phi_1=\int_0^\sigma\sqrt{1+b(1+2\epsilon b\psi_0)^{-1}}\,d\sigma \tag{10c}$$

The quadrature ϕ_1 must be evaluated numerically following the computation of ψ_0.

The complete outer solution to first order in ϵ is

$$\lambda=b(1+2\epsilon b\psi_0)^{-1} \tag{11a}$$

$$\omega=e^{-\phi_1} \tag{11b}$$

Clearly, flight-path angle (or λ) is constant to zeroth-order since gravity is a first-order effect. Gravity causes λ to decrease. With decreasing altitude (increasing σ), horizontal velocity (or ω) decays exponentially owing to drag.

The transformation from the nondimensional (ω, λ) variables to the relative velocity components (v,w) results in

$$v^2=V^2\omega=V^2e^{-\phi_1} \tag{12a}$$

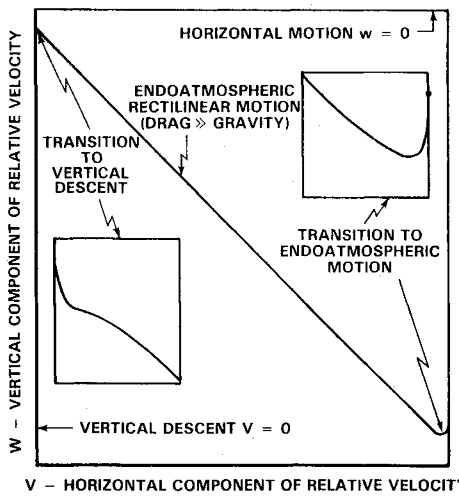

Fig. 1 Schematic of a typical entry trajectory in (v,w) velocity space.

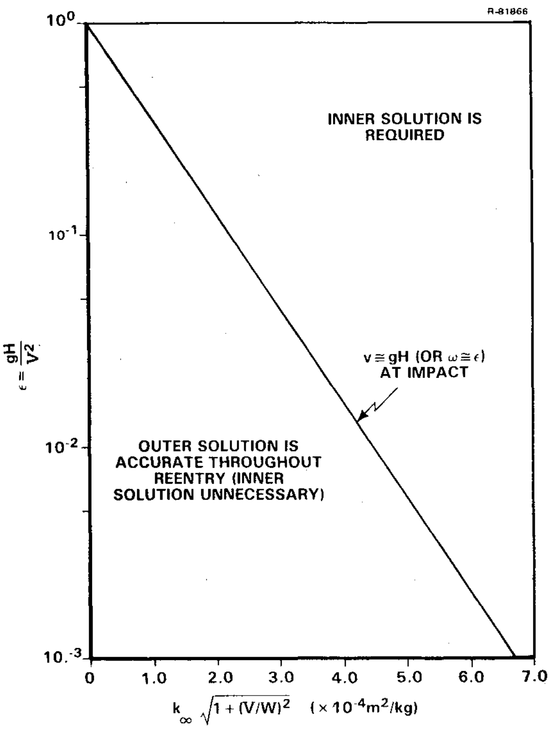

Fig. 2 Criteria that determine if the inner solution is necessary.

$$w^2 = v^2/\lambda = \left[W^2 + 2g \int_h^H e^{\sqrt{1+b\sigma}} \mathrm{d}h \right] e^{-\phi_1} \tag{12b}$$

where W is the vertical relative velocity component at the entry point. If $k_\infty \to 0$, the outer solution reduces to

$$\lim_{k_\infty \to 0} \{v^2\} = V^2 = \text{const} \tag{13a}$$

$$\lim_{k_\infty \to 0} \{w^2\} = W^2 + 2g(H-h) \tag{13b}$$

which are recognized as the closed-form solutions for velocity as a function of altitude on a (flat Earth) parabolic trajectory. If $V \to 0$, the outer solution reduces to

$$\lim_{V \to 0} \{v^2\} = 0 \tag{14a}$$

$$\lim_{V \to 0} \{w^2\} = \left[W^2 + 2g \int_h^H e^{\sigma} \mathrm{d}h \right] e^{-\sigma} \tag{14b}$$

which is the known solution for vertical descent.[1] Finally, if u denotes the instantaneous relative velocity magnitude, it follows from Eqs. (12) that:

$$u^2 = \left[V^2 + 2g \int_h^H e^{\sqrt{1+b\sigma}} \mathrm{d}h \right] e^{-\phi_1} \tag{15}$$

where U is the entry point value of u. If $k_\infty \to 0$, Eq. (15) reduces to the energy integral in a uniform gravity field.

The flat earth simplification is inaccurate for describing entry from circular orbit ($b=0$). It is clear that Eq. (5b) is singular in this case. Slow changes in flight path angle occur if $\mathrm{d}\lambda/\mathrm{d}\zeta < 1$ at the entry point, or if $2\epsilon b^2 < 1$, which is equivalent to

$$(W \tan\Gamma)^2 > 2gH$$

where Γ is the flight-path angle at the entry point. Since it was assumed that λ changes slowly, the inequality Eq. (15) defines an admissible set of initial conditions for which the outer solution may be used.

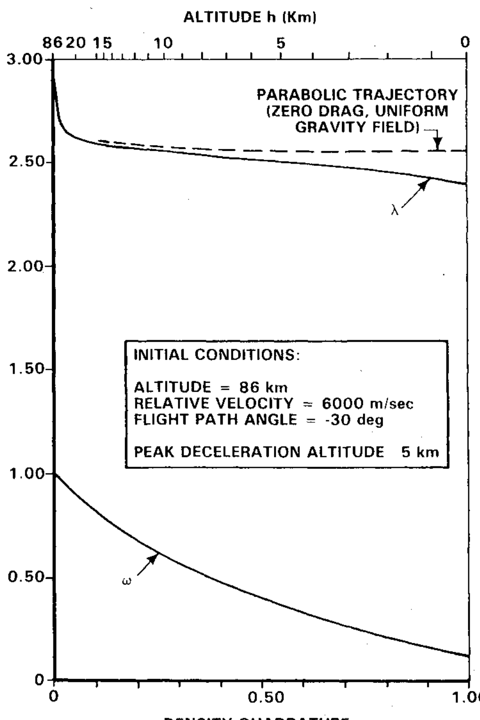

Fig. 3 Theoretical results for $k_\infty = 1 \times 10^{-4}$ m²/kg.

The Inner Solution

The transition to vertical descent is characterized by rapid changes in the flight path angle (or λ). Rapid changes in λ will begin to occur when $\omega = 0(\epsilon)$, or $v^2 = gH$, since a small quantity then multiplies the highest-order derivative in Eq. (5b). Rather than include many higher-order correction terms in the outer solution, it is more efficient to introduce an inner solution, accurate for $\omega < \epsilon$. The inner limit of the outer solution, which is also the outer limit of the inner solution, is the altitude h^* at which $\omega = \epsilon$. To zeroth-order, $\omega = \epsilon$ if [refer to Eq. (10a)]

$$|\ell n \epsilon| = \sqrt{1+b\sigma^*} = \sqrt{1+b} k_\infty \int_{h^*}^H \rho \mathrm{d}h \tag{16}$$

which determines h^*.

The initial conditions (V, W) and the aerodynamic area-to-mass ratio k_∞ are parameters that determine if an inner solution is necessary. The transition to vertical descent cannot occur if $\omega > \epsilon$ at impact, and the outer solution will then predict λ accurately throughout entry. If Eq. (16) is evaluated at $h^* = 0$, a locus is defined that divides the parameter space (Fig. 2) into two regions. For example, if $\epsilon = 0.03$ and $b = 3.0$, the inner solution is unnecessary if $k_\infty < 1.70 \times 10^{-4}$ m²/kg. Existing analytic theories of ballistic entry motion[9-11] are generally most accurate in this region of the parameter space.

The inner solution will be derived using a new set of dependent variables Ω and Λ, defined by

$$\Omega = \omega/\epsilon \qquad \Lambda = \lambda \tag{17}$$

A straightforward substitution of Eq. (17) in Eq. (5) results in

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\sigma} + \sqrt{1+\Lambda}\Omega = 0 \tag{18a}$$

$$\Omega \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{\Lambda}\right) + 2 = 0 \tag{18b}$$

Dilation of ω removes the small quantity multiplying the highest-order derivative in Eq. (5b). The absence of $0(\epsilon)$ terms in Eq. (18b) shows that gravity causes significant changes in flight-path angle (or λ). The independent variable ζ is *not* dilated, and the density quadrature σ is unaffected.

The solutions of the differential equations (18) may be written in the form

$$\Omega = e^{-\Phi} \tag{19a}$$

$$\Lambda = B(1+2B\Psi)^{-1} \tag{19b}$$

The quadratures Φ and Ψ satisfy a set of simultaneous integral equations

$$\Phi = \int_{\sigma^*}^{\sigma} \sqrt{1+\Lambda} \mathrm{d}\sigma = \int_{\sigma^*}^{\sigma} \sqrt{1+B(1+2B\Psi)^{-1}} \mathrm{d}\sigma \tag{20}$$

$$\Psi = \int_{\zeta}^{\zeta^*} \frac{\mathrm{d}\zeta}{\Omega} = \int_{\zeta}^{\zeta^*} e^{\Phi} \mathrm{d}\zeta \tag{21}$$

The constant B is determined by evaluating the outer solution at $\zeta = \zeta^*$.

The method of successive approximations will be used to solve for Φ and Ψ. Since ζ^* is typically $0(\epsilon)$ and $\zeta \leq \zeta^*$, it follows from Eq. (21) that Ψ is also a small quantity. The zeroth-order approximation is derived assuming $\Psi = 0$, from which it follows:

$$\Phi_0 = \int_{\sigma^*}^{\sigma} \sqrt{1+B} \mathrm{d}\sigma = \sqrt{1+B}(\sigma - \sigma^*) \tag{22a}$$

Table 1 1976 Standard atmosphere density model

Layer l	Base density ρ_l, kg/m^3	Base altitude h_l, km	Scale height H_l, km	Exponent b_l
1	1.2250×10^0	0	−44.331	−4.256
2	3.6392×10^{-1}	11	6.342	...
3	8.8035×10^{-2}	20	216.650	35.163
4	1.3225×10^{-2}	32	81.661	13.201
5	1.4275×10^{-3}	47	7.922	...
6	8.6160×10^{-4}	51	−96.661	−11.201
7	6.4211×10^{-5}	71	−107.325	−16.082

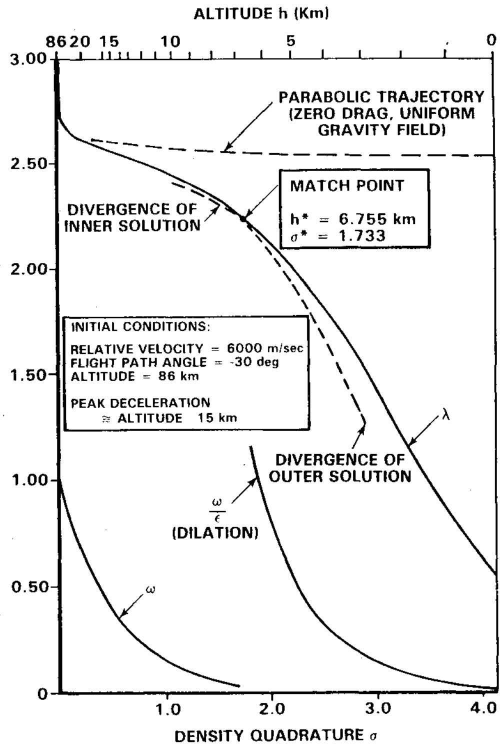

Fig. 4 Theoretical results for $k_\infty = 4\times 10^{-4}$ m^2/kg.

$$\Psi_0 = \int_\zeta^{\zeta^*} e^{\sqrt{1+B(\sigma-\sigma^*)}}\,\mathrm{d}\zeta \tag{22b}$$

The quadrature Φ_0 has a closed-form solution, but Ψ_0 must be evaluated numerically. The zeroth-order quadratures are used to generate

$$\Phi_1 = \int_{\sigma^*}^{\sigma} \sqrt{1+B(1+2B\Psi_0)^{-1}}\mathrm{d}\sigma \tag{23a}$$

$$\Psi_1 = \int_\zeta^{\zeta^*} e^{\Phi_1}\,\mathrm{d}\zeta \tag{23b}$$

Both quadratures are computed numerically.

The complete inner solution is

$$\Omega = e^{-\Phi_1} \tag{24a}$$

$$\Lambda = B(1+2B\Psi_1)^{-1} \tag{24b}$$

The constant B is determined using Eq. (11b):

$$B = V^2\left[W^2 + 2g\int_{h^*}^{H} e^{\sqrt{1+b\sigma}}\mathrm{d}h\right]^{-1} \tag{24c}$$

The transformation from the nondimensional (Ω,Λ) variables to the relative velocity components (v,w) results in

$$v^2 = V^2\epsilon\Omega = gHe^{-\Phi_1} \tag{25a}$$

$$w^2 = V^2\Lambda = gH\left[\frac{1}{B} + 2\Psi_1\right]e^{-\Phi_1} \tag{25b}$$

Results

Numerical results will be computed using the multilayer 1976 Standard Atmosphere model.[16] A different $\rho(h)$ function is used in each layer, depending on its temperature profile. In adiabatic layers, the model assumes that temperature decreases (or increases) linearly with h, and a power-law density function is used:

$$\rho(h) = \rho_l[1+(h-h_l)/H_l]^{-b_l} \tag{26a}$$

The constants ρ_l, h_l, H_l, and b_l are atmosphere model parameters for layer l (Table 1). In isothermal layers, temperature is constant, and an exponential density function is used:

$$\rho(h) = \rho_l\exp\{-(h-h_l)/H_l\} \tag{26b}$$

At the interfaces between each of the seven layers, ρ is continuous but $\mathrm{d}\rho/\mathrm{d}h$ is discontinuous. The top of the seventh layer, at $h_8 = H = 86$ km, is taken to be the entry point.

Since the density functions, Eqs. (26), are integrable, the density quadrature σ is a closed-form function of altitude. It is evaluated piecewise across each layer of the 1976 Standard Atmosphere model. For example, if $h_1 \le h \le h_2$, σ is given by

$$\sigma(h) = k_\infty\left[\int_h^{h_2}\rho\mathrm{d}h + \sum_{l=2}^{7}\int_{h_l}^{h_{l+1}}\rho\mathrm{d}h\right] \tag{27a}$$

In layer l, the integrated density profiles for the adiabatic and isothermal layers, respectively, are given by

$$\int_h^{h_{l+1}}\rho\mathrm{d}h = \frac{(H_l+h-h_l)\rho(h)}{1-b_l}\Bigg|_h^{h_{l+1}} \tag{27b}$$

$$\int_h^{h_{l+1}}\rho\mathrm{d}h = -H_l\rho(h)\Big|_h^{h_{l+1}} \tag{27c}$$

where it is understood that $h_l \le h \le h_{l+1}$. In particular, the numerical value of $\sigma(0)/k_\infty$ at impact is 10,332 kg/m^2.

The altitude histories of the nondimensional variables ω and λ are displayed in Figs. 3 and 4. For convenience, relative velocity magnitude u, flight-path angle γ, horizontal component v, and vertical component w are obtained using

$$u = V\sqrt{\omega(1+1/\lambda)} \tag{28a}$$

$$\gamma = -\tan^{-1}(\sqrt{1/\lambda}) \tag{28b}$$

$$v = V\sqrt{\omega} \tag{28c}$$

$$w = V\sqrt{\omega/\lambda} \tag{28d}$$

where $V = 5196$ m/s for the initial conditions shown. The outer solution is accurate during the entire entry phase if $k_\infty = 1 \times 10^{-4}$ m²/kg† (Fig. 3). An inner solution is needed if $k_\infty = 4 \times 10^{-4}$ m²/kg (Fig. 4). In this case, theoretical results differ from numerical results (flat Earth trajectories) by $\delta v = +4$ m/s and $\delta\gamma = +0.2$ deg at impact.

During the initial entry phase, the trajectory is approximately parabolic. Since it is unaffected by drag, γ changes only slightly due to gravity prior to peak deceleration. Drag causes a significant reduction in v. The total change in γ is larger with drag included, compared to gravity acting alone, due to the reduction in v. It then follows from Eq. (2b) that the total horizontal distance traveled in a uniform gravity field is always reduced by drag.

The inner solution is initialized when $\omega = \epsilon$ (Fig. 4), typically after peak deceleration occurs. Since v is much smaller compared to the zero drag case, significant changes in trajectory curvature (or λ) occur due to gravity. The flight path approaches a vertical asymptote as v, which is unaffected by gravity, decays monotonically due to drag. Ultimately, vertical descent ($\lambda = 0$ or $\gamma = -90$ deg) can occur near impact if, for example, $k_\infty > 4 \times 10^{-4}$ m²/kg.

Concluding Remarks

A new semianalytic solution of the ballistic entry problem accurately described several important types of entry motion. In particular, the transition to vertical descent was accurately predicted by a new inner solution. A number of important simplifying assumptions that were made in the development should be reevaluated in the future, as follows.

The solution was formulated using a *constant* aerodynamic area-to-mass ratio. A more realistic drag coefficient model should include Mach number dependence arising from inviscid flow compressibility effects. For example, the inviscid drag coefficient can increase significantly as speed decreases and Mach number approaches unity. As drag coefficient increases, speed decreases exponentially, and significant modifications in the inner solution are anticipated. The drag model should also include viscous drag, lift-induced drag (for lifting vehicles), and ablation-induced shape change and mass loss.

The essential features of ballistic entry dynamics were included in the simplified equations of motion. In-plane and lateral forces arising from aerodynamic lift and local wind perturbations can be particularly significant at low velocities and should be included in the inner solution. Although the gravity field and kinematic models are simpler for a flat Earth, the more general spherical Earth formulation must be used to describe entry from circular orbit.

†For convenience, this value corresponds to a ballistic coefficient value 2048 lb/ft².

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